Classical Mechanics of Spinning Patricle in a Curved Space

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Abstract

An example of mechanical system whose configuration space is direct product of a curved space and the local group of rotations, is presented. The system is considered as a model of spinning particle moving in the space. The Hamiltonian formalism for this system and possible method for its quantization are discussed. It is shown that the Hamilton equations coincide with the Papapetrou equations for spinning test-particle in general relativity.

1 Introduction

Motion of spinning objects in curved spaces has been studied by numerous authors [1, 2, 3, 4] as a purely general relativistic problem. However, one special case of this phenomenon has been described exhaustively in standard texts on classical mechanics. In order to show this consider the simplest case of such a motion assuming that the space is a two-dimensional sphere and the moving object is a two-dimensional rigid body covering a domain in this sphere. As the sphere is a symmetric space and, hence, it admits free motions of a rigid body in it, the body represents a kind of spinning objects moving in this space. On the other hand, this mechanical system imbedded into three-dimensional Euclidean space constitutes a case of three-dimensional rigid body with fixed point. Indeed, the center of the sphere can be considered as a point the body because all the distances between this and other points of the body remain constant thus, inclusion of the center does not break the rigidity condition. Consequently, the well-known branch of mechanics studying motions of rigid bodies constitutes a primer of mechanics of rotating objects in curved spaces. In the present work we compose non-relativistic Hamiltonian description of mechanical systems formed by spinning objects moving in an arbitrary curved space.

The first problem arising when considering motions of a rigid body in an arbitrary curved space is that, in general, such a space does not admit free motions of a rigid body covering some domain on it. In order to eliminate this difficulty we suggest some generalization of the conventional definition of rigid body. Due to our definition a rigid body with center of mass in some point x of the space covers a domain in the tangent space T_x over this point [5] and therefore we will call it "tangent rigid body". Apparently, such a body is able to move freely on a curved space regardless of intrinsic geometry of this space.

Below we will distinguish two main kinds of motion of tangent rigid body. One of them is motion of the center of mass under which the body obeys the law of parralel transfer. Another one represents pure rotation, i.e., the motion under which the center of mass rests. As was pointed out in the work [4] the configuration space of the system is locally diffeomorfic to the group of symmetry of the tangent space (the Poincare group for the case considered in the cited work). Therefore, its

subspaces containing trajectories of the main motions, are orthogonal. The orthogonality means that the kinetic energy of the system is equal to the sum of kinetic energies of the main motions. Nevertheless, these orthogonal motions depend on each other and their interaction yields a difference between behaviours of tangent rigid bodies and mass points.

In order to demonstrate this interaction consider the following special case of tangent rigid body. Let the space will be specified as a surface in three-dimensional Euclidean space and the body is a plate remaining tangent to the surface. Then, the vector of angular momentum of the body, concerned with its rotation, lies on the normal to the surface established in the center of mass of the body. Since any infinitesimal displacement of the body yields an infinitesimal rotation of the vector of angular momentum it yields a moment of force acting on the body. In the simplest cases the moment of force is collinear with the displacement. The constraints confining the body in tangent planes transform this moment into a force damping body tranversally to the original displacement. Consequently, the force acting on a body depends linearly on its velocity, angular momentum and the space curature, whereas its direction is orthogonal to the velocity. Below we generalize these consideratons to the case of arbitrary Riemann space.

2 Definition of tangent rigid body

Consider a Riemann space M endowed with a coordinate system $\{x^i\}$ and let 1-forms θ^a

$$\theta^a = e^a_i(x)dx^i \tag{1}$$

constitute an orthonormal frame in every point of the space M such that its metric tensor may be expressed through the Kronecker delta:

$$g = g_{ij}dx^i \otimes dx^j = \delta_{ab}\theta^a \otimes \theta^b.$$

The connection 1-form for these frames may be introduced through the first structure equation:

$$d\theta^a = \omega^a_b \wedge \theta^b, \quad \omega_{ab} + \omega_{ba} = 0 \tag{2}$$

and the connection coefficients γ_{abc} are that of expansion of this 1-form in the local frames $\{\theta^a\}$:

$$\omega^c_{b.} = \gamma^c_{ab.}\theta^a. \tag{3}$$

The Riemann curvature is represented by the 2-form $\Omega^b_{a.}$ defined by the second structure equation

$$\Omega^b_{a.} = d\omega^b_{a.} + \omega^c_{a.} \wedge \omega^b_{c.} \tag{4}$$

and the coefficients of its expansion in the local frame of 2-forms $\{\theta^a \wedge \theta^b\}$ constitute the components of the Riemann tensor of curvature:

$$\Omega^b_{a.} = R^b_{cda.}\theta^c \wedge \theta^d. \tag{5}$$

By analogy with classical definition of rigid body [6] the notion of tangent rigid body may be defined as a collection of mass points moving in the tangent space under holonomy constraints due to which the distance between any two of them remains constant. Thus, let T_x be a tangent space

over a point x, m_α and $r_\alpha \in T_x$ be masses and radius-vectors of the points forming the body. Then, apparently, the mass of the body is equal to

$$m = \sum_\alpha m_\alpha \tag{6}$$

Assuming that the center of mass of the body coincides with the point $x \in M$ one obtains the following identity:

$$\sum_\alpha m_\alpha r_\alpha = 0. \tag{7}$$

The following sum represents components of the tensor of inertia of the body:

$$I^{ab} = \sum_\alpha m_\alpha r^a_\alpha r^b_\alpha \tag{8}$$

and it will be assumed below that the body is symmetric such that this tensor has the form of Kronecker delta:

$$I^{ab} = I\delta^{ab}. \tag{9}$$

As well as in the classical case [6] the configuration space of the tangent rigid body is the direct product of space M and the group of rotations of local frames. Coordinates of the system in this space are specified by coordinates of the center of mass x^i and by orientation of the body with respect to the local frame $\{\,\theta^a\}$. Since none of forces act in this system its Lagrangian and Hamiltonian are equal to its kinetic energy [6]. Now, in order to compose Lagrangian and Hamiltonian formalisms for this system we introduce generalized velocities concerned with coordinates of the system in the configuration space.

3 Generalized coordinares and generalized velocities

As was pointed out above the generalized coordinates of the system in question are coordinates x^i of the body center of mass and some functions of components of the vectors r_α referred to the local frame. Consequently, the corresponding generalized velocities are $\dot{x}^i$ and some functions of components of the vectors r_α and their time derivatives $\frac{d}{dt}r^a_\alpha$, depending on the derivatives linearly. The rigidity conditions require existance of antisymmetric tensor $\eta^{b}_{a.}$ of angular velocity specifying all the time derivatives of r^a_α's as follows:

$$\frac{d}{dt}r^a_\alpha = \eta^a_{b.} r^b_\alpha.$$

The components of this tensor may be considered as generalized velocities of the system. Since the kinetic energy of the body does not depend on its orientation the corresponding generalized coordinates may be omitted such that the Lagrangian is a function of variables x^i, $\dot{x}^i$ and $\eta^b_{a.}$ only. It must be pointed out that definition ofangular velocities introduced above do not account changes of local frames along the center of mass trajectory. Therefore, it is necessary to introduce the covariant time derivative of r_α components:

$$\dot{r}^a_\alpha \equiv (\eta^a_{b.} + \dot{x}^c \gamma^a_{cb.}) r^b_\alpha. \tag{10}$$

In order to compose the expression of Lagrangian consider an infinitesimally small tangent rigid body assuming that all the mass points forming the body lie in a small neighbourhood of the center of mass and have coordinates $x + r_\alpha$. Then their velocities are $\dot{x} + \dot{r}_\alpha$ with $\dot{r}_\alpha$ being defined in the equality (10). Now, taking into account the equations (6-10) one obtains the following expression for kinetic energy:

$$L = T = \frac{1}{2}\sum_\alpha m_\alpha(\dot{x} + \dot{r}_\alpha, \dot{x} + \dot{r}_\alpha) = \frac{m}{2}(\dot{x}, \dot{x}) + \frac{1}{2}I(\eta^a_{a.} + \gamma^b_{cb.}\dot{x}^c)(\eta^b_{a.} + \gamma^b_{c.a}\dot{x}^c) \tag{11}$$

This expression conveys only the first non-vanishing approximation for the kinetic energy. In this case the further analysis is related to the system defined by the obtained form of Lagrangian. It must be pointed out that non-orthogonality of this form does not contradict the above statement that the main kinds of motion are orthogonal, because, by definition, the angular velocities $\eta^b_{a.}$ are non-covariant.

4 Generalized momenta and Poisson brackets

The direct evaluation of the generalized momenta from the expression(11) yields

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = [(m\delta_{ab} + I\gamma^d_{ac.}\gamma^c_{a.d})\dot{x}^b + I\gamma_{abc}\eta^{bc}]e^a_i; \quad S_{ab} = \frac{\partial L}{\partial \eta^{ab}} = I(\eta_{ab} + \gamma_{cab}\dot{x}^c) \tag{12}$$

Here e^a_i is the matrix introduced in the equation (1), p_a are components of the body momentum and S_{ab} are that of its spin. The Poisson brackets for the spin components coincide with commutators of corresponding generators of the group of rotations:

$$[S_{ab}, S_{cd}] = \frac{1}{2}(\delta_{ac}S_{bd} + \delta_{bd}S_{ac} - \delta_{ad}S_{bc} - \delta_{bc}S_{ad}) \tag{13}$$

Other Poisson brackets for generalized momenta are, by definition, equal to zero:

$$[p_i, p_j] = [p_i, S_{ab}] = 0 \tag{14}$$

whereas $[p_a, p_b]$ is not zero since the orthonormal components p_a contain the factors e^i_a depending on x^i's.

In order to obtain the explicit form of the Hamiltonian one must express generalized velocities through the generalized momenta. The direct evaluation yields:

$$I\eta_{ab} = S_{ab} - \gamma_{cab}\dot{x}^c.$$

Substituting this into the first equality (12) one obtains:

$$m\dot{x}^a = \delta^{ab}(p_b - \gamma_{bcd}S^{cd}). \tag{15}$$

The last two results transform the expression (11) to the following form:

$$H = \frac{1}{2m}\delta^{ab}(p_a - \gamma_{acd}S^{cd})(p_b - \gamma_{bcd}S^{cd}) + \frac{S^2}{2I} \tag{16}$$

5 Equations of motion

As follows from the second equation (12) and independence of the Lagrangian on the body orientation the spin S_{ab} remains constant on tha trajectory. Consequently, the last term in the equation (16) is constant and, hence, it can be substracted from the Hamiltonian, i.e., the Hamiltonian can be written in the form

$$H = \frac{1}{2m}\delta^{ab}(p_a - \gamma_{acd}S^{cd})(p_b - \gamma_{bcd}S^{cd}) \tag{17}$$

It is to be pointed out that the equality (15) is conspicuously similar to the well-known definition of the covariant momentum of a particle moving in a gauge field and the expression (17) coincides with that of Hamiltonian of such a particle. Remarkable, the field in this analogy is represented by connection coefficients, whereas the particle spin represents its charge coupled to the field.

Derivations of the Hamilton equation for spin leads, with account of the Poisson brackets (13) and the equation (15), to the following equality:

$$\frac{dS_{ab}}{dt} = [H, S_{ab}] = -\dot{x}^c\gamma_c^{de}[S_{de}, S_{ab}] = -\dot{x}^c(\gamma^{d}_{ca.}S_{db} + \gamma^{d}_{cb.}S_{ad}),$$

or, in agreement with the Papapetrou's result,

$$\frac{DS_{ab}}{dt} = 0 \tag{18}$$

Now, it remains to analyze the Hamilton equation

$$\frac{dp_a}{dt} = [H, p_a] \tag{19}$$

It must be pointed out that in the case of zero spin this equation coincides with that of geodesic and, consequently, all its spin-independent terms constitute the covariant time derivative of the vector $m\dot{x}$. The left hand side of this equation is

$$\frac{dp_a}{dt} = m\delta_{ab}\frac{d\dot{x}^b}{dt} + \dot{x}^d\partial_d\gamma_{abc}S^{bc} + \gamma_{abc}\frac{dS^{bc}}{dt}$$

where the derivatives ∂_a are defined as follows:

$$\partial_a = e^i_a\frac{\partial}{\partial x^i}.$$

Substituting here the time derivative of S^{bc} from thr equation (18) one finds that

$$\frac{dp_a}{dt} = m\delta_{ab}\frac{d\dot{x}^b}{dt} + \dot{x}^d\gamma_{abc}(\gamma^{b}_{d.e}S^{ec} + \gamma^{c}_{d.e}S^{be}).$$

The right-hand side of the equation (19) contains the only spin-depending term

$$\dot{x}^d[p_a, \gamma_{dbc}]S^{bc} = (\partial_a\gamma_{dbc})\dot{x}^dS^{bc}.$$

Now, collecting all the evaluated terms and taking into account the fact that all the spin-independent terms yield the covariant time derivative of $m\dot{x}$ and the definitions of connection and curvature (2-5) one finds that the Hamilton equation (19) coincides with the Papapetrou equation [1]

$$m\frac{D\dot{x}^a}{dt} = R^a_{.bcd}\dot{x}^bS^{cd}.$$

6 Remarks on quantization

Quantization of the mechanical system considered above changes it into a quantum mechanical description of motion of a particle with spin in the space M. Operators of the spin components $\hat{S}_{ab}$ form a representation of the Lie algebra of the group of rotations and, hence, the wave function takes its values in the vector space of this representation. Now, chosing the operators of the particle momentum componenets in accord with the Poisson brackets (14)

$$p_i = \frac{1}{\imath}\frac{\partial}{\partial x^i}$$

one finds that the Hamiltonian operator contains only covariant derivatives of the wave function:

$$\hat{H} = g^{ij}(\frac{1}{\imath}\frac{\partial}{\partial x^i} + e_i^a \gamma_{abc}\hat{S}^{bc})(\frac{1}{\imath}\frac{\partial}{\partial x^j} + e_j^a \gamma_{abc}\hat{S}^{bc}).$$

In other words, the particle momentum is represented by the operator of covariant derivative

$$\hat{P}_i = \frac{1}{\imath}\frac{\partial}{\partial x^i} + e_i^a \gamma_{abc}\hat{S}^{bc}$$

instead of the operator of generalized momentum introduced above.

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